Analisis of Shot Noise at Finite Temperatures in FQH Edge States

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We investigate shot noise at *finite temperatures* induced by the quasi-particle tunneling between fractional quantum Hall (FQH) edge states. The resulting Fano factor has the peak structure at a certain bias voltage. Such a structure indicates that quasi-particles are weakly *glued* due to thermal fluctuation. We show that the effect makes it possible to probe the difference of statistics between $\nu = 1/5$, $2/5$ FQH states where quasi-particles have the same unit charge. Finally we propose a way to indirectly obtain statistical angle in hierarchical FQH states.

KEYWORDS: fractional quantum Hall effect, Non-equilibrium Kubo formula, shot noise

The fractional quantum Hall (FQH) effect occurs in the two dimensional electron system subject to a strong magnetic field. The Hall conductance exhibits plateaus at filling fractions.[1] What is most surprising is that the phenomena is explained by existence of the quasi-particle with a fractional charge e^*.[2] The FQH state becomes an incompressible fluid with an energy gap. A confining geometry in low disorder samples makes the edges gapless modes, which carries fractionally charged particles. Thus direct observation of e^* at edge states has been a stimulating problem.

As the pioneering work Kane and Fisher proposed the setup to measure shot noise associated with the tunneling between counter-propagating edge states narrowed at a quantum point contact (QPC).[3] It was shown that the Poisson noise at zero temperature is proportional to e^* which arises from the granularity of charge. Inspired by the result, $e^* = e/3$ of the Laughlin quasi-particle was directly observed.[4, 5] The applicability has enhanced further studies of $e^* = e/5$ in a hierarchical FQH state with $\nu = 2/5$.[6] The well-established scheme is applied to various filling factors, for example $\nu = 5/2$[7] and $\nu = 2/3$.[8]

It is also predicted that the quasi-particle obeys fractional statistics,[9] characterized by a statistical angle θ upon an adiabatic exchange process of two quasi-particles. In two spatial dimensions, a statistical angle is allowed to take an intermediate one between $\theta = 0$ for boson and $\theta = \pi$ for fermion, namely *anyons*. The issue to observe θ has also become an intriguing question.

To detect θ, on the basis of quantum interference, it was proposed to use a kind of Aharanov-Bohm (AB) effect in equilibrium.[10–14] The recent studies succeeded to detect the relative statistics.[15] Other attempts were made in Ref.[16], and Refs.[17,18] where cross-correlation in three edge states was studied for the Laughlin states at zero temperature. Kim extended their works into hierarchical FQH states at finite frequency and temperature, and then discussed statistics for $\nu = 1/5$, $2/5$.[19] It is well known that quasi-particles in $\nu = 1/5$, $2/5$ FQH states have the same charge, but obey different statistics. Shot noise measurement in the low-temperature limit succeeded to confirm the former,[6] but failed to do the

latter. Ref.[19] gave an idea to the problem, but not realized in experiments.

In this paper, we revert back to the standard set up,[3–8] and discuss shot noise in two edge states instead of three ones. What is most important point is that finite temperature effects are considered on the basis of our recent work.[20] It is shown that the approach enables us to detect the difference of statistics in $\nu = 1/5$, $2/5$ FQH states. Finally we discuss a method to determine statistical angle itself in hierarchical FQH states.

First of all, let us start with our formalism: shot noise at *finite temperatures* based on the nonequilibrium Kubo formula in mesoscopic systems.[20] It is well known that the usual Kubo formula determines linear conductance. In contrast, we derived a relation to differential conductance G under finite bias voltages. Thus this formula was called as the nonequilibrium Kubo formula. Then, it was also proposed to define shot noise at any temperature S_h as the following formula:

$$S_h \equiv -\left\langle \{\delta I_B, e^*(\delta N_L - \delta N_R)\} \right\rangle, \qquad (1)$$

where $\delta A \equiv A - \langle A \rangle$. I_B and $N_{L,R}$ are the backscattering current and number of quasi-particles in left/right reservoirs. It was proved that S_h in eq.(1) has several aspects expected as shot noise. (i) In a non-interacting system, S_h directly gives the Landauer-type shot noise at finite temperatures. (ii) At zero temperature, S_h is agreement with the standard shot noise: current noise S at $T = 0$. (iii) In the linear response regime, $S_h = 0$ and eq.(2) reproduces the Nyquist-Johnson relation. As a result S_h is qualified as shot noise at finite temperatures. Actually using S_h it was successful to study shot noise of the Kondo effect in a quantum dot.[21, 22]

The nonequilibrium Kubo formula also satisfies the relation:

$$S_h = S - 4k_B T G, \qquad (2)$$

where S is current noise, $G = \partial_V I_B$ is differential conductance, k_B is Boltzmann constant, and T is temperature in reservoirs. This relation shows us what variations of thermal noise should be subtracted from current noise to define shot noise at finite temperatures in experiments. Our framework gives a prospective way to study shot

noise at finite temperatures: S_h in eq.(1) is directly calculated, and its prediction is examined through $S - 4k_BTG$ in experiments. In the following the approach is applied to edge states.

Let us consider the quasi-particle tunneling through the QPC set at $x = 0$ between edge states.[3,23,24] The gapless edge states are described by chiral Tomonaga-Luttinger liquids. We begin with Laughlin states with filling fractions $\nu = 1/(2n+1)$ for simplicity. The Hamiltonian is given by right/left going edge modes H_R, H_L and the tunneling part H_B: $H = H_R + H_L + H_B$,

$$H_{R,L} = \frac{v_F}{\pi} \int_{-\infty}^{\infty} dx \left(\frac{\partial \phi_{R,L}(x)}{\partial x} \right)^2, \tag{3}$$

$$H_B = t_B \psi_R^{\dagger}(0)\psi_L(0) + h.c., \tag{4}$$

where we put Planck constant $\hbar$ one, v_F is the Fermi velocity, t_B is the tunneling amplitude of quasi-particles, $\phi_{R,L}(x)$ are chiral boson fields, and $\psi_{R,L}(x)$ represent the vertex operators in $c = 1$ CFT as

$$\psi_{R,L}(x) = \frac{1}{\sqrt{2\pi}} e^{\pm ik_F x} : e^{\mp i\sqrt{\nu}\phi_{R,L}(x)} :. \tag{5}$$

The quasi-particle hopping with an unit charge e^* generates backscattering current:

$$\hat{I}_B \equiv ie^* \left(e^{i\omega_0 t} t_B \psi_R^{\dagger}(0)\psi_L(0) - e^{-i\omega_0 t} t_B^* \psi_L^{\dagger}(0)\psi_R(0) \right). \tag{6}$$

Following the discussion of a gauge transformation,[24] source-drain bias voltage V is incorporated into phase factor $\omega_0 = e^*V$. The backscattering current and current noise are obtained as

$$I_B = \left\langle \hat{I}_B(t) \right\rangle, \qquad S = \int dt' \left\langle \left\{ \hat{I}_B(t), \hat{I}_B(t') \right\} \right\rangle.$$

These quantities can be calculated on the basis of Schwinger-Keldysh formalism.

Conventionally information on a fractional charge is extracted from Poisson noise. Thus, when the quasi-particle weakly transmits through the QPC, it is sufficient to calculate these transport quantities up to the lowest order $\mathcal{O}(t_B^2)$. These expressions are rewritten into Landauer forms:

$$I_B = e^* \int \frac{d\omega}{2\pi} T(\omega)(f_L - f_R), \tag{7}$$

$$S = e^{*2} \int \frac{d\omega}{\pi} T(\omega)(f_L(1 - f_L) + f_R(1 - f_R))$$

$$+ e^{*2} \int \frac{d\omega}{\pi} T(\omega)(f_L - f_R)^2. \tag{8}$$

The transmission probability is characterized by the t-matrix $t(\omega) \equiv \pi t_B \rho(\omega)$:

$$T(\omega) \equiv t^*(\omega - \omega_0/2) t(\omega + \omega_0/2), \tag{9}$$

$$t(\omega) = \frac{t_B}{v_F} \cosh\left(\frac{\beta\omega}{2}\right) \left(\frac{2\pi}{\beta v_F}\right)^{\nu-1} \frac{|\Gamma\left(\frac{\nu}{2} + i\frac{\beta\omega}{2\pi}\right)|^2}{\pi\Gamma(\nu)}.$$

Here $\rho(\omega) \equiv -\mathrm{Im}G^r(\omega)/\pi$ is the density of states (DOS)

at $x = 0$. The quasi-particle Green function is defined by

$$G_{R,L}^{\alpha\beta}(t) \equiv -i \left\langle T_K \psi_{R,L}(t^{\alpha}) \psi_{R,L}^{\dagger}(0^{\beta}) \right\rangle,$$

$$G_{R,L}^{\pm\mp}(t) = \frac{\pm i}{2\pi} \left(\frac{i\frac{\pi}{\beta}}{\sinh\frac{\pi}{\beta}(t \pm i\epsilon)} \right)^{\nu}, \tag{10}$$

where ϵ is an infinitesimal positive number where $\alpha, \beta = \pm$ represents a branch of the Keldysh contour and the retarded component becomes $\mathrm{Im}G^r = i(G^{+-} - G^{-+})/2$.

In the $\nu = 1$ IQH state or non-interacting edge states, $T(\omega)$ in eq.(9) leads to a constant $|t_B|^2$. Within our approximation, only the transmission probability is renormalized by Coulomb interaction through the DOS. In contrast the Fermi distribution function is unrenormalized as $f_{L,R} \equiv \frac{1}{1+\exp(\beta(\omega \pm \omega_0/2))}$.

Seemingly, the second line in eq.(8) might be interpreted as shot noise in view of a Landauer-formula sense. However, shot noise formula in eq.(1) modifies the naive prediction. Following the same approximation, eq.(1) can be calculated, and rewritten into a Landauer-like form:

$$S_h = S_L + \delta S_L, \tag{11}$$

$$S_L = e^{*2} \int \frac{d\omega}{\pi} T(\omega)(f_L - f_R)^2,$$

$$\delta S_L = e^{*2} \int \frac{d\omega}{\pi} T(\omega)(f_L - f_R)(y_L - y_R), \tag{12}$$

where S_L represents a Landauer-type shot noise and δS_L does the correction term. Here,

$$y(\omega) \equiv \frac{1}{2} \tanh\left(\frac{\beta\omega}{2}\right) - \frac{1}{\pi} \mathrm{Im}\left[\Psi\left(\frac{\nu}{2} + i\frac{\beta\omega}{2\pi}\right) \right],$$

$$y_{L,R} \equiv y(\omega \pm \omega_0/2), \tag{13}$$

where $\Psi(z)$ is the digamma function. In case of $\nu = 1$, because $\delta S_L = 0$ and $T(\omega) = |t_B|^2$, it is exemplified that S_h is equivalent to the Landauer-type shot noise S_L for $\nu = 1$ at finite temperatures. Thus the correction term δS_L plays an essential role in FQH states.

Let us discuss the feature in view of the nonequilibrium Kubo formula. $G = \partial_V I_B$ is calculated using eq.(7), and then the resulting G and S are substituted into $S - 4k_BTG$. Therefore, we confirm that the result is identical with S_h in eq.(11), so that the nonequilibrium Kubo formula eq.(2) is satisfied. In the context it is found that δS_L corresponds to the V-derivative of $T(\omega)$.

To proceed a further discussion, we introduce the following Fano factor:

$$F_\nu \equiv \frac{S_h}{2e^* I_B}. \tag{14}$$

The different point compared to a standard Fano factor is to be normalized by an unit charge e^*. In the low-temperature/high-bias limit, $S_h/2I_B$ converges to e^*, as discussed later. With the normalization factor at zero temperature, it enables us to focus on thermal fluctuation of shot noise. Here in Fig.1 F_ν for $\nu = 1/3$ at a fixed inverse temperature $\beta = 1$ is compared to $F_{L\nu}$ and $\delta F_{L\nu}$ defined by $F_{L\nu} \equiv \frac{S_L}{2e^* I_B}$, $\delta F_{L\nu} \equiv \frac{\delta S_L}{2e^* I_B}$. $F_{L\nu}$ monotonously changes, on the other hand $\delta F_{L\nu}$ exhibits
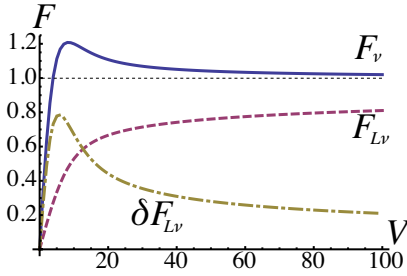
Fig. 1. Bias dependence of F_ν (solid line), $F_{L\nu}$ (dashed line) and $\delta F_{L\nu}$ (dashed-dotted line) for $\nu = 1/3$ at $\beta = 1$
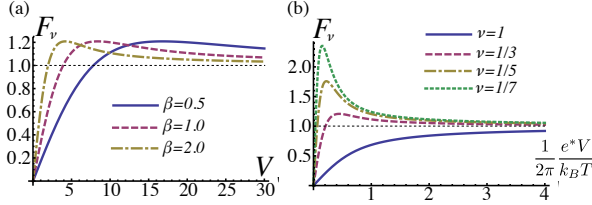

Fig. 2. (a) F_ν for $\beta = 0.5, 1.0, 2.0$ at $\nu = 1/3$ (solid line, dashed line, dash-dotted line); (b) F_ν for $\nu = 1, 1/3, 1/5, 1/7$ at $\beta = 1$ (solid line, dashed line, dash-dotted line, dotted line)

non-monotonousness with increasing bias voltage. The total Fano factor F_ν which is the sum of them converges to 1 in the high-bias limit. The fact represents that the charge of the quasiparticle is e^* because F_ν is normalized by the unit charge. In contrast, we find the peak structure at a finite bias which originates from the correction noise $\delta F_{L\nu}$. It turns out that the enhancement from the unit charge occurs. Therefore, transmitted quasi-particles tend to come together due to thermal fluctuation.

The peak structure is a sign for carried charges to bunch induced by thermal fluctuation. Therefore we call the effect "*thermal bunching*". Note that this "*thermal bunching*" is different from the bunching which originates from statistics of quasi-particles.[19]

Fig.2(a) shows F_ν at a fixed $\nu = 1/3$ for several temperatures β. As lowering temperature, a peak position moves to a lower bias. What is universal nature is that the peak height is unchanged. On the other hand in Fig.2(b) F_ν at a fixed $\beta = 1$ is drawn for some parameters ν. The peak structure does not appear in the $\nu = 1$ IQH state. The peak height becomes larger for smaller ν, namely, stronger magnetic field or Coulomb interaction.

Up to now, our discussion has been restricted to Laughlin states with $\nu = 1/(2n + 1)$. As said in the introduction, we would like to consider statistics for $\nu = 1/5,\ 2/5$. Here thus we extend our discussion into hierarchical FQH states (ex. $\nu = 2/5, 3/7, 2/9, \cdots$). Those states are characterized by filling fraction, unit charge

and statistical angle:

$$\nu = \frac{p}{2np + 1} \quad e^* = e\frac{1}{2np + 1} \quad \theta = \pi\frac{2n(p - 1) + 1}{2np + 1} \tag{15}$$

which are originally defined through the K-matrix.[23] Our formalism developed above is described by the quasi-particle Green function at $x = 0$. This treatment can be also justified when multiple tunneling processes can be neglected(discuss later). Thus the extension changes the exponent ν to α in eq.(10):

$$\alpha = \frac{1}{p(2np + 1)}. \tag{16}$$

Furthermore, the current eq.(7), current noise eq.(8) and shot noise eq.(11) have been expressed as a frequency-integral form, in accordance with the concept of Landauer formula. Concerning the current and current noise, integrated results were derived for Laughlin states.[24] According to the same idea, shot noise can be also calculated. The result is straightforwardly extended into hierarchical FQH states, and thus the Fano factor is governed by a scaling function:

$$F_\alpha = \frac{2}{\pi}\mathrm{Im}\left[\Psi\left(\alpha + i\frac{1}{2\pi}\left(\frac{e^*V}{k_BT}\right)\right)\right]. \tag{17}$$

This function is characterized by exponent α. If we define Fano factor as a ratio between total current noise S and backscattering current I_B, it only gives $\coth(\omega_0\beta/2)$, which does not include α.

Taking advantage of the scaling form, let us reexamine the peak structure of the Fano factor. We plot $\partial_V F_\alpha$ taking α as a continuous parameter. It is found that $\alpha < 1/2$ is the sufficient condition for emergence of peak. It is easily shown that $\alpha = 1/p(2np + 1)$ is less than $1/2$, and thus in all type of hierarchical FQH states a peak structure develops.

As an experimentally relevant case, let us discuss statistics in FQH states with $\nu = 1/5,\ 2/5$. The quantities listed in eq.(15) are specifically obtained for these states:

$\nu = 1/5$	$e^*_{1/5} = e/5$	$\theta_{1/5} = \pi/5$	$(n, p) = (2, 1)$
$\nu = 2/5$	$e^*_{2/5} = e/5$	$\theta_{2/5} = 3\pi/5$	$(n, p) = (1, 2)$

As mentioned in the introduction, $e^*_{1/5} = e^*_{2/5} = e/5$ has been confirmed through shot noise measurement in the low-temperature limit.[6] However, there still remain the problem on statistics.

To address the issue we begin with generic relations among e^*, θ, ν and α in eqs.(15) and (16). Each of e^*, θ, ν and α is determined by two integer parameters: n and p. Thus the independent quantities become two of them, and others are given by them. Therefore the statistical angle discussed here is represented in all type of hierarchical FQH states as

$$\theta = \pi\left[1 - \alpha\left(\frac{e}{e^*} - 1\right)\right]. \tag{18}$$

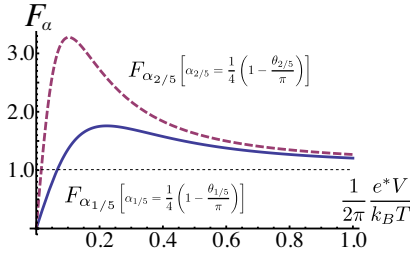The result yields one-on-one relations between α and θ

Fig. 3. Fano factors of $\nu = 1/5$ (Laughlin state) and $\nu = 2/5$ (hierarchical FQH state). Both quasi-particles have the same fractional charge $e/5$.

using $e^*_{1/5} = e^*_{2/5} = e/5$:

$$\alpha_{1/5} = \frac{1}{4}\left(1 - \frac{\theta_{1/5}}{\pi}\right), \quad \alpha_{2/5} = \frac{1}{4}\left(1 - \frac{\theta_{2/5}}{\pi}\right). \quad (19)$$

In conclusion, in order to see the difference of statistical angles there is a way to discuss exponents α.

We substitute $\alpha_{1/5}$, $\alpha_{2/5}$ in eq.(19) into eq.(17) for $\nu = 1/5$, $2/5$ respectively, and show the Fano factors for $\nu = 1/5$, $2/5$ in Fig.(3). In the low-temperature/high-bias limit, both Fano factors converge to 1. This is a generic feature of the Fano factor normalized by unit charge e^*. In the present case for $\nu = 1/5$, $2/5$, even if the normalization by e is considered, the limiting values are equal: $e^*_{1/5}/e = e^*_{2/5}/e = 1/5$. It turns out that shot noise in the low-temperature/high-bias limit cannot distinguish statistics. What is striking is that the Fano factors of $\nu = 1/5, 2/5$ have difference at finite temperature/bias. The Fano factor is determined by observable quantities thruogh eq.(2) and (14). Analyzing its Fano factor, it is possible to distinguish statistics of $\theta_{1/5}$ and $\theta_{2/5}$ in experiments.

Having studied on specific statistics for $\nu = 1/5, 2/5$, the last discussion turns to issue to determine statistical angles of general hierarchical FQH states. As pointed out in eq.(18), statistical angle θ is given by unit charge e^* and exponent α. It is well established that an unit charge e^* is determined through shot noise in the low-temperature limit. Note that exponent α is also available in the following. By measuring the Fano factor F_α, and then fitting the data to the scaling function in eq.(17) as a parameter α, one can infer the exponent α. Thus we suggest an alternative procedure to determine θ by extracting e^* and α from shot noise experiments. The estimation of exponent α has been already reported by analyzing the power-law dependence of tunneling current: $I \propto V^\alpha$.[25] Our approach makes it possible to obtain both α and e^* in the shot noise measurement with the scaling function eq.(17) .

Finally we comment on closely-related works. Ferarro et al.[26] pointed out that the tunneling is dominated by multiple particles at $T < T^*$, in contrast the single particle at $T > T^*$. The dynamics of neutral edge mode determines the crossover temperature T^*, evaluated as 50mK in the experiment.[27] Thus our treatment, which has focused on the single-particle process, still stands in the region of $T > T^*$. As another work, Isakov et al.[16] showed

a monotonic-behavior of Fano factor in non-interacting particles with exclusion statistics. Thus the interaction between particles is relevant for "thermal bunching".

In summary the finite-temperature shot noise at FQH edge states has been studied on the basis of the nonequilibrium Kubo formula. The peak structure of Fano factor has been found at a bias voltage. We have named the phenomena "thermal bunching" because this is a sign for quasi-particles to weakly glue, mediated by thermal fluctuation. The phenomena has been determined by a scaling function characterized by an exponent of quasi-particle Green function. In $\nu = 1/5$, $2/5$ FQH states, exponents have been given by only statistical angles. Detecting the discrepancy of Fano factors, one can measure the difference of statistics. Finally we have proposed an indirect way to determine a statistical angle from exponent fitted to the scaling function and unit charge estimated at sufficiently low temperature within $T > T^*$

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